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TRANSIENT RADIATION LOADING OF A SPHERICAL PULSED SONAR ARRAY.(U)
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CAMBRIDGE ACOUSTICAL ASSOCIATES, INC.

LEVEL II

RESEARCH, DEVELOPMENT AND CONSULTING IN ENGINEERING AND SCIENCE

TRANSIENT RADIATION LOADING OF A SPHERICAL PULSED SONAR ARRAY

J. M. Garrelick and M. C. Junger

January 1970

Final Report U-335-219
Prepared for Computer Applications, Inc.
Systems Engineering Division
Contract N00024-69-C-1342, CAI-SED 7105-03

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LEVEL II

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J. M. Garrelick [redacted] M. C. Junger

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Abstract

The transient pressure on a spherical array arising from the excitation of a simple circular element has been obtained. The cases of a Heaviside step and a continuous wave (CW) pulse were considered. The low frequency, i.e., long time, response was obtained via a classical wave harmonic analysis. High frequency asymptotic expansions and the Watson transformation were used to isolate the short time transient effects into efficient "creeping wave" series. Dispersion, caused by the spherical curvature, is shown to produce attenuation of the traveling wave and theoretically an exponentially decaying pulse of infinite duration. These creeping wave series are not valid in the immediate vicinity of, and on, the active element. However, in this region the Kirchhoff approximation has been shown to be applicable.

List of Symbols

a	radius of sphere
b	radius of piston in plane baffle
c_m	$= \lvert \alpha_m' \rvert / 2^{4/3}$
c	sound velocity in acoustic medium
G	Green's function
$H(t)$	Unit or Heaviside step function
h_m	spherical hankel function of the first kind of order m
h_m'	derivative of spherical hankel function of the first kind of order m
k	ω/c
k_o	ω_o/c
P_m	Legendre polynomial
p	acoustic pressure
R,θ,ψ	spherical coordinates
r	radius vector measured from center of piston in plane baffle
s	Laplace transform variable
t	time
t_o	time of maximum pressure
v	velocity
ρ	density of acoustic medium
α	one-half angle subtended by piston in spherical baffle
α_m'	root of the derivative of the Airy function
w	circular frequency
w_o	circular frequency of CW pulse
$\sim$	indicates Fourier or Laplace transform

1. Introduction

The spatial fluctuations of surface pressure on a large array give rise to a surface pressure distribution whose average over individual transducer elements varies widely from one transducer element to another, if these elements are small in terms of wavelengths. Consequently, unless the differences in radiation loading are electrically or mechanically compensated, the dynamic configuration of the array and hence the sound field will not be as specified by the beam-forming calculations. This difficulty does not arise if elements have characteristic dimensions of one-half wavelength or more because the fluctuations in surface pressure amplitude tend to average out when integrated over the surface of an element. The need of compensating for surface pressure fluctuations is therefore primarily associated with the long wavelengths resulting from the recent trend toward long-range and hence low-frequency echo-ranging.

The need and implementation of compensating for the variation in the radiation loading has been thoroughly analyzed[1]. For this purpose the radiation impedance matrix, i.e. the surface pressure averaged over individual transducer elements when a single transducer element is active, must be computed.

This study provides an analytical solution suitable for computing the transient radiation loading on a spherical BQS-6 type array at the onset of a step or a continuous-wave (CW) pulse, with no restriction as to beam steering and formation. Preliminary numerical results were presented at the Fall 1969 meeting of the Acoustical Society of America in San Diego (paper 4B1).

The required surface pressure can be analyzed by means of the classical wave harmonic formulation[2]. However this series exhibits poor convergence for large arrays in terms of wave length, particularly in the vicinity of the transient wave fronts.

An alternative approach, which is undertaken in this work, had originally been developed by Watson[3] to compute the diffraction, around the earth, of electromagnetic waves from a point source. That formulation has recently been applied to the evaluation of the surface pressure on large sonar arrays[4,5]. This "creeping-wave" formulation is tantamount to an expansion of the pressure in negative powers of ka and is therefore particularly well suited to large arrays. Unfortunately, it does not yield the self-impedance, i.e., the pressure on the active element. However, in the large-ka range where Watson's technique is useful, the self-impedance can be approximated by using the self-impedance of a plane piston in a plane baffle, with a minor correction applied for baffle curvature to the resistive component. For the ka and transducer element dimension of the BQS-6, the resulting correction to the self-impedance is shown to be negligible.

2. Mathematical Model

Consider the transient radiation loading associated with a single, active element located on a spherical array when:

(A) the active element undergoes a velocity step

$$\dot{w}(t) = \dot{W}H(t) \tag{1}$$

(B) the active element initiates a "CW-pulse"

$$\dot{w}(t) = \dot{W}H(t) \sin \omega_o t \tag{2}$$

The transient surface pressures thus constitute the transient equivalent of the familiar radiation impedance matrix. Item (A) constitutes an extension to a spherical baffle, of Miles analysis of a piston in a plane baffle[6].

Item (B) constitutes a similar extension of Mangulis' analysis[7].

For the BQS-6 the array is large in terms of the CW-wavelengths (e.g. $ka = 32$), and the active element is small (e.g. a linear dimension of $14\lambda/45$, corresponding to a subtended angle of 3.5^{o}).

Under these conditions, the surface pressures can be computed over most of the sphere as though the active element were a point source. This assumption is not valid in the immediate vicinity of, and on the active element. The spherical polar axis ($\theta = 0$) is made to coincide with the center of the active element (Figure 1). The active element subtends the angle

$$0 \leq \theta \leq \alpha$$

The mathematical problem may therefore be stated as the solution to the wave equation in spherical coordinates

$$\nabla^2_{sph}\, p + k^2 p = 0 \tag{3}$$

subject to the boundary condition

$$\dot{w}\,(a,\theta;t) = f\,(t)\; H\,(\alpha - \theta) \tag{4}$$

where $f\,(t)$ is defined by equation (1) or (2) and $k = \omega/c$

3. Wave-Harmonic Analysis

The transient radiation loading can be formulated in terms of the inverse Fourier transform of the transient velocity distribution

$$p\,(a,\theta;t) = \sum_{n=0}^{\infty} A_n\, P_n\,(\cos\theta) \int_{-\infty}^{\infty} \tilde{\dot{w}}\,(\omega)\, \frac{h_n(ka)}{h_n'(ka)}\, e^{-i\omega t}\, d\omega \tag{5}$$

Expanding this velocity distribution in Legendre functions[8] the prescribed boundary condition becomes

$$\dot{w}\,(a,\theta;t) = f\,(t) \sum_{n=0}^{\infty} [P_{n-1}\,(\cos\,\alpha) - P_{n+1}\,(\cos\,\alpha)]\; P_n\,(\cos\,\theta) \tag{6}$$

where f (t) is defined by equation (1) or (2). If the piston radius is allowed to tend to zero,

$$\dot{w}\,(\theta_o;t_o) \approx \frac{1}{2}\;\alpha^2\,f\,(t) \sum_{n=0} (n + \tfrac{1}{2})\;P_n\,(\cos\,\theta_o),\; \alpha^2 \ll n^{-2} \tag{7}$$

For any finite piston, however small, the higher order terms must eventually be written in the form of Equation 6, when n becomes large enough. This unfortunately, will be shown to be insufficient to insure convergence of the surface pressure at the beginning of the pulse.

The Fourier transform of the pressure therefore, is a series in the form

$$\tilde{p}(a,\theta;\omega) = -\,\frac{i\rho c}{2\pi}\;\tilde{f}\;(\omega) \sum_{n=0} [P_{n-1}(\cos\,\alpha)-P_{n+1}(\cos\,\alpha)]\;\frac{h_n(ka)}{h_n'(ka)}\;P_n(\cos\,\theta) \tag{8}$$

The corresponding inverse transform can be evaluated using residue theory which expresses the pressure as series of wave-harmonics. The transient component of the solution is associated with the residues arising from the roots of $h_n'(ka) = 0$. The number of roots[9] and hence of residues associated with the n^{th} wave-harmonic is $(n + 1)$. Any steady-state contribution is produced by the singularities of $\tilde{f}\;(\omega)$. The transient pressure field thus becomes:

For case (A)

$$p(a,\theta;t) = i\rho c\dot{W}H(t) \sum_{n=0} \sum_{j=1}^{n+1} \frac{h_n(z_{nj})\exp(-iz_{nj}ct/a)}{z_{nj}h_n''(z_{nj})}\;X$$

$$[P_{n-1}(\cos\,\alpha) - P_{n+1}(\cos\,\alpha)]P_n(\cos\,\theta) \tag{9}$$

For case (B)

$$p(a,\theta;t)=\rho c\dot{W}H(t)\sum_{n=0}\left\{\sum_{j=1}^{n+1}\frac{k_oa}{z_{nj}^2-(k_oa)^2}\,\frac{h_n(z_{nj})e^{-iz_{nj}ct/a}}{h_n''(z_{nj})}+\frac{1}{2(k_oa)}\,\frac{h_n(k_oa)}{h_n'(k_oa)}\,e^{-i(k_oa)\frac{ct}{a}}\right\}$$

$$[P_{n-1}(\cos\alpha)-P_{n+1}(\cos\alpha)]\;P_n(\cos\theta) \qquad (10)$$

where z_{nj} are the complex roots of the equation $h_n'(ka) = 0$ and $k_o = \omega_o/c$. The first term of equation (10) represents the transient effect. The second term arises from a pole at $\omega = \omega_o$ and is the steady-state contribution.

Consider the convergence properties of the residue series, Equation 9 and 10. The surface pressure is, by virtue of the Initial Value Theorem

$$\lim_{t\to 0} p(a,\theta;t) = -\lim_{\omega\to\infty} i\omega p(a,\theta;\omega) \qquad (11)$$

Letting $k \to \infty$ in Equation 8, the ratios h_n/h_n' tend to i. Substituting in the above asymptotic relation, the early-time surface pressure becomes proportional to

$$\lim_{t\to 0} p(a,\theta;t) \propto \sum_{n=0} [P_{n-1}(\cos\alpha)-P_{n+1}(\cos\alpha)]\;P_n(\cos\theta) \qquad (12)$$

This series converges no faster than the series representing the velocity distribution Equations 6 and 7. The wave-harmonic series is therefore not a practical formulation for evaluating the pulse at early times particularly for small pistons, for which Equation 7 applies to a large number of terms. It thus becomes apparent that the early-time transient radiation loading on a piston set in a spherical baffle is not tractable by means of the rigorous

wave-harmonic formulation described above. This is in fact true for any values of the (θ,t) domain corresponding to the vicinity of the propagating transient wave front. This is in contrast to the circular piston, whose double-transform solution yields the radiation loading at early times without difficulty.

Asymptotic techniques are therefore required. In the next sections an approximate solution is developed for obtaining the transient pressure on the array specialized to the situation of slowest convergence, i.e., the limiting case of the small piston. Furthermore, the solution is deliberately specialized to early times, i.e., to the high-frequency end of the spectrum. This is achieved by applying the Watson transformation to the pressure transform, in Equation 8, specialized to small α.

4. Creeping-Wave Formulation - Velocity Step

The following analysis uses Watson's transformation to replace the steady-state wave-harmonic series with a residue or "creeping-wave" series which captures the high-frequency components of the solution in its leading terms. The resulting transform takes a form suitable for integration by steepest-descent approximation. One thus achieves not only rapid convergence, but one has cast the integrand in a form which permits analytical evaluation of the inverse Fourier transform.

This technique has been validated for the CW surface pressure generated on a large cylinder by a point source[10]. In this case, the pressure is a Fourier transform over all axial wavenumbers, rather than over frequency, but the same asymptotic integration applies.

Consider case (A), i.e., the velocity step. The formal expression for the transient surface pressure specialized to infinitesimal pistons is obtained by using Equation 7, in lieu of Equation 6, for the modal velocity

coefficients. Thus using the small piston approximation to expand the spatial dependence of Equation 4, i.e. Equation 7, and the Fourier transform of Equation 1, Equation 5 becomes:

$$p(a,\theta;t) = -\frac{\rho c\, \alpha^2}{4\pi}\dot{W}\int_{-\infty}^{\infty}\frac{e^{-i\omega t}}{\omega}\sum_{n=0}(n+\tfrac{1}{2})\frac{h_n(ka)}{h_n'(ka)}P_n(\cos\theta)\,d\omega \tag{13}$$

Note that the integrand of Equation 13 apparently contains a singularity at $\omega = o$. However, since

$$\underset{x\to 0}{\text{Limit}}\ \frac{h_n(x)}{h_n'(x)} = O(x)$$

the integrand is finite at $\omega = 0$ and in fact no singularity exists. The wave-harmonic series in the above expression can be written as Watson's residue series:

$$\sum_{n=0}(n+\tfrac{1}{2})\frac{h_n(x)}{h_n'(x)}P_n(\cos\theta) \simeq -\frac{x^{1/2}}{2^{1/6}}\,x^{7/6}\exp(-i\pi/12\ \ X$$
$$\sum_{m=0}\frac{1}{|\alpha_m'|}\left\{\frac{e^{i\nu_m\theta}}{(\sin\theta)^{1/2}} + \frac{e^{i\nu_m(2\pi-\theta)}}{[\sin(2\pi\ \theta)]^{1/2}}\right\}\ \theta \neq 0,\ \pi \tag{14}$$

$$\simeq -\ 2^{1/3}\pi\ x^{5/3}\exp(-i\pi/3)\sum_{m=0}\frac{e^{i\nu_m\pi}}{|\alpha_m'|},\ \theta = \pi \tag{15}$$

where

$$x \equiv ka$$
$$\nu_m = x + |\alpha_m'|\ (-x/2)^{1/3} \tag{16}$$

$-\ |\alpha_m'|$ = roots of the derivative of the Airy function[9]

$= 1.02,\ 3.25,\ 4.82,\ \ldots\ [3\pi(4m+1)/8]^{2/3}$

The rapid convergence of the creeping-wave series for large x, i.e., at high dimensionless frequencies ka, is apparent if the exponent is split into its real and imaginary components, thus displaying an exponentially decaying factor:

$$e^{i\nu_m \theta} = \exp\left[i\left(x + \frac{x^{1/3}|\alpha_m'|}{2^{4/3}}\right)\theta \right] \exp\left[-\frac{x^{1/3}\, 3^{1/2}|\alpha_m'|}{2^{4/3}}\theta \right]$$

It is precisely this good convergence of the series at higher frequencies which makes it suitable for exploring the early times of a transient event.

The creeping-wave series does not converge on, or in the immediate vicinity of, the active element. Fortunately, in the high-frequency limit where this formulation is useful, the surface pressure in this region can be approximated by ignoring baffle curvature as $\theta \to \alpha$. The steady-state surface pressure field generated by a piston on a plane baffle blends smoothly with the field computed from the creeping-wave series[10]. Similarly, under transient conditions, the surface pressure on the active element will be shown to be approximated at early times by the solution derived for the plane piston (section 7).

In the early-time, high-frequency limit, the $(2\pi - \theta)$ term in Equation 14 can be dropped, except in the immediate velocity of the antipodes $\theta = \pi$. Thus, retaining only the θ - term and substituting the resulting creeping-wave series in Equation 13, the transient surface pressure becomes

$$p(a,\theta;t) = \frac{\rho\, \dot{W}\alpha^2\, a^{7/6}\, e^{-i\pi/12}}{8(2\pi \sin\theta)^{\frac{1}{2}}\, c^{1/6}} \sum_{m=0} \frac{1}{c_m} \int_{-\infty}^{\infty} \omega^{1/6} \exp(-i\omega t + i\nu_m \theta)\, d\omega, \quad \theta \neq 0, \pi \tag{17}$$

Since ν_m which is of order $\omega a/c$, is large in the high-frequency limit the phase angle is caused to vary rapidly with ω. The inverse transform in Equation 17 thus lends itself to integration by the method of steepest decent

. . . . whereby an integral of the form

$$I = \int_{-\infty}^{\infty} \Phi(\omega)\, e^{i\psi(\omega)}\, d\omega \tag{18}$$

can be approximated by the contribution of the saddle-point $\bar{\omega}$ defined by the condition

$$\frac{d\psi}{d\omega} = 0, \quad \omega = \bar{\omega} \tag{19}$$

This yields

$$I \simeq \frac{(2\pi i)^{\frac{1}{2}}\, \Phi(\bar{\omega})\, e^{i\psi(\bar{\omega})}}{[\psi''(\bar{\omega})]^{\frac{1}{2}}} \tag{20}$$

Using Equation 17, Equation 19 yields

$$\frac{\partial \psi_m}{\partial\, \omega} = -\, t + \theta\, \frac{\partial \nu_m}{\partial\, \omega} = 0, \quad \omega = \bar{\omega}_m \tag{21}$$

From Equation 16

$$\frac{\partial \nu_m}{\partial\, \omega} = \frac{a}{c} + \frac{|\alpha'_m|}{3}\, \frac{(-a)^{1/3}}{(2c\omega^2)^{1/3}} \tag{22}$$

When this is substituted in Equation 21, one can solve for the saddle-point value, $\omega = \bar{\omega}_m$

$$\bar{\omega}_m = \frac{(|\alpha'_m|\theta \quad /3)^{3/2}}{[t-(a\theta/c)]^{3/2}}\, (-a/2c)^{\frac{1}{2}} \tag{23}$$

As $[t-(a\ \theta/c)]\to 0$, $\bar{\omega}_m \to \infty$. The condition $t \to (a\ \theta/c)$ therefore represents a wave front spreading over the sphere from $\theta = 0$, by virtue of the Initial Value Theorem. The stationary-phase approximation enveloping the spherical

baffle integral must therefore be multiplied by the coefficient H $[t-(a\ \theta/c)]$.

The modulus of the integrand in Equation 17 becomes

$$\Phi(\bar{\omega}_m) = \frac{(|\alpha_m'|\ \theta/3)^{1/4}}{[t-(a\ \theta/c)]^{1/4}} (a/2c)^{1/12}\ e^{i\pi/12} \tag{24}$$

The second derivative of the phase ang   is also required in constructing Equation 20

$$\begin{aligned}\frac{\partial^2 \psi_m}{\partial\omega^2} &= \theta\ \frac{\partial^2 \nu_m}{\partial\omega^2} \\ &= \frac{2^{2/3}\ |\alpha_m'|\theta}{9}\ \frac{(-a)^{1/3}}{(c\omega^5)^{1/3}} \\ &= \frac{2(6c)^{1/2}[t-(a\ \theta/c)]^{15/6} e^{i\pi/2}}{a^{1/2}\ (|\alpha_m'|\theta)^{3/2}}\ ,\ \omega = \bar{\omega}_m\end{aligned} \tag{25}$$

Finally, the phase angle evaluated at $\omega = \bar{\omega}_m$ is required:

$$\begin{aligned}\psi_m(\omega) &= -\ \omega(t-\frac{a\theta}{c}) + |\alpha_m'|\ \theta\ (-\ \frac{\omega a}{2c})^{1/3} \\ &= \frac{i(2a)^{1/2}(|\alpha_m'|\theta)^{3/2}}{3^{3/2}\ c^{1/2}[t-(a\theta/c)]^{1/2}}\ ,\ \omega = \bar{\omega}_m\end{aligned} \tag{26}$$

The phase angle is purely imaginary, thus giving rise to an exponentially decaying term. When the results in Equation 24, 25, and 26 are substituted in Equation 20, the stationary-phase approximation to the integrals in Equation 17 is obtained:

$$\int_{-\infty}^{\infty} d\omega = \frac{\pi^{1/2}\left|\alpha_m'\right|\theta a^{1/3}\, e^{-i\pi/12}}{3^{1/2}(2c)^{1/3}[t-(a\theta/c)]^{3/2}} \exp\left\{-\frac{(2a)^{1/2}(\left|\alpha_m'\right|\theta)^{3/2}}{3^{3/2}c^{1/2}[t-(a\theta/c)]^{1/2}}\right\} H(t-\frac{a\theta}{c}) \quad (27)$$

When this result is substituted in Equation 17, and expressing $|\alpha_m'|$ in terms of the coefficients $c_m \equiv |\alpha_m'|/2^{4/3}$, the expression for the transient pressure finally becomes:

$$p(a,\theta;t) = \frac{\rho\dot{W}\,\alpha^2 a^{3/2}\theta\, H[t-(a\theta/c)]}{4(6c\,\sin\theta)^{1/2}\,[t-(a\theta/c)]^{3/2}} \sum_{m=0} \exp\left\{-\frac{4(2a)^{1/2}\,(c_m\theta)^{3/2}}{3^{3/2}c^{1/2}\,[t-(a\theta/c)]^{1/2}}\right\} \quad (28)$$

This solution is not applicable at or near $\theta = 0$. Here it may supplement

the solution for the impulsively accelerated plane piston derived by Miles[6] and reproduced in section 6.

It was mentioned earlier that the creeping-wave solution does not converge in the illuminated region $\alpha > \theta$. Actually, the validity of the solution is more generally limited in the region near $\theta = 0$ and $\theta = \pi$ because it embodies a large $|\nu \sin\theta|$ approximation to the hypergeometric function $P_{\nu-1/2}$, a function which arises from P_n as a result of the Watson transformation. Consequently, the creeping-wave solution predicts, incorrectly, a surface pressure larger than the plane baffle solution in a region bounded approximately by $\theta < 50\,\alpha^{2/5}$, with both θ and α in degrees. For a comparable active element, this region is far wider than for the steady-state situation. In the transient situation, the α-dependence of the extent of this region arises because the creeping-wave solution which predicts a pressure varying linearly with the volume velocity $\dot{Q}$ ($\propto \dot{W}\alpha^2$), is smaller than the small-piston (or large-range) limit for the plane baffle, which varies as $\dot{W}b$, i.e. as $\dot{Q}/b$,

and therefore becomes singular as $b \to 0$. This difference arises because for field points on a curved baffle, the acoustic shadow cast by the baffle compounds the spherical spreading loss thus eliminating the singularity associated with a point source. For a plane baffle, where the shadow effect is absent spherical spreading alone mitigates but does not eliminate this singularity. On the active element, a singularity of the form $\dot{Q}/b^2$ arises.

Another significant difference between the two baffle configurations arises from the dispersive and attenuated character of the creeping-waves: For plane baffles the pressure pulse displays, like the piston velocity, a time-derivative which is infinite at the leading edge. For the curved baffle, the onset of the pulse represented by Equation 28 is gradual, all time derivatives being zero near the leading edge, as expected from the fact that exponential attenuation increases monotonically with frequency thus extinguishing the signal associated with the leading edge of the pulse. Furthermore, while the pulse duration $T = 2b/c$ tends to zero for the plane baffle as the source dimension $2b$ tends to zero, this is not the case for the curved baffle. The reason is the dispersive character of the creeping-wave which lengthens the pulse duration as the pulse envelops the sphere. The time at which the maximum of a given creeping-wave mode occurs, obtained by setting the time derivative of Equation 28 equal to zero, is

$$t_m = \frac{a\theta}{c} + \frac{2a(|\alpha_m'|\theta)^3}{3^5\, c} \tag{29}$$

Since, for all cases of interest, only the lowest creeping-wave mode is required at the pressure peak, one obtains an analytical expression for

the pressure maximum:

$$\frac{p(a,\theta;t_o)}{\rho\, c\dot{W}\, \alpha^2} = \frac{3^7\, e^{-3}}{2^4 \left|\alpha_o'\right|^{9/2} (\theta^7 \sin\theta)^{1/2}} = \frac{6.20}{(\theta^7 \sin\theta)^{1/2}} \tag{30}$$

Thus, the further the angular distance traveled by the pulse, the more the peak pressure falls behind the wavefront. Furthermore until the caustic at $\theta = \pi$ is approached the spreading loss far exceeds the spherical spreading loss observed on the plane baffle. A final comment on the solution is that, even though a high-frequency asymptotic technique has been used, the solution tends to the correct late-time limit, viz. zero.

The result in Equation 28 coincides with the solution derived by Friedlander[11] using a different formulation. His Green's Function approach is described in the next section and used to obtain the transient pressures at the beginning of a CW-pulse. Even though Friedlander's asymptotic approximations and method of integration are identical to those above, his procedure differs in that he does not apply Watson's transformation to the wave-harmonic series, but finds a solution in the form of a creeping-wave series from the wave equation.

5. Green's Function Formulation - CW-Pulse

The transient pressure expressed as an inverse Fourier transform (Equation 5) can be written in terms of the steady-state Green's function, specialized to surface pressures ($R = a$), and axisymmetric velocity distributions. This Green's function will be designated as $\tilde{G}$, to distinguish it from the transient or impulse Green's function G used by Friedlander. The steady-state pressure will be similarly expressed as $\tilde{p}$, as will the displacement distribution $\tilde{W}(\theta;\omega)$

of the boundary. The CW surface pressure can now be written as

$$\tilde{p}(a,\theta;\omega) = 2\pi\,\rho\omega^2 a^2 \int_0^{\pi} \tilde{G}(a,\theta_o;\omega)\,\tilde{w}(\theta_o;\omega)\sin\theta_o\,d\theta_o$$
$$\approx -\pi\,\rho\alpha^2 a^2\,\tilde{G}(a,\theta|a,0;\omega)\,\tilde{W}(\omega),\ \alpha << 1 \tag{31}$$

The latter approximation is not necessary, but simplifies the analysis without impairing the results in the region where the creeping-wave solution is efficient.

Noting that Equation 5 is of the form

$$p(a,\theta;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{p(a,\theta\ \omega)}{\tilde{W}(\omega)}\,\tilde{W}(\omega)\,e^{-\ \omega t}\,d\omega \tag{32}$$

and expressing the ratio $\tilde{p}/\tilde{W}$ in terms of Equation 31, the transient pressure becomes

$$p(a,\theta;t) = -\ \frac{\rho\alpha^2 a^2}{2}\int_{-\infty}^{\infty} \tilde{G}(a,\theta|a,0;\omega)\,\tilde{W}(\omega)\,e^{-i\omega t}\,d\omega \tag{33}$$

For the velocity history in Equation 2; i.e., the CW-pulse,

$$W(t) = \hat{W}\,\omega_o\,H(t)\cos\omega_o t \tag{34}$$

whose Fourier transform is

$$\tilde{W}(\omega) = \frac{i\,\dot{W}\,\omega_o\omega}{\omega^2 - \omega_o^2} \tag{35}$$

When this is substituted in Equation 33, the transient pressure becomes

$$p(a,\theta;t) = -\frac{\rho\alpha^2 a^2}{2} i\dot{W}\omega_o \int_{-\infty}^{\infty} \frac{\omega}{\omega^2 - \omega_o^2} \tilde{G}(a,\theta|a,0;\omega)\, e^{-i\omega t}\, d\omega \qquad (36)$$

The resulting pressure in this situation is made up of two contributions to the inverse transform. A saddle-point contribution which represents the transient effect will be shown to tend to zero with increasing time. The residue at $\omega = \omega_o$ is not attenuated with time and in fact is the steady-state creeping-wave solution.

A.) Steady-State Response

The steady-state problem of a point source on a spherical radiator has previously been solved[10]. For the sake of continuity, the results are presented below

$$p(a,\theta) = \frac{\rho c^2 Q(ka)^{13/6}}{(32\pi \sin\theta)^{1/2} a^3} \sum_{m=0}^{\infty} \frac{e^{[-\alpha_m (ka)^{1/3}\theta + i\Phi_m(\theta)]}}{c_m} \qquad \theta \neq 0,\ \pi \qquad (37)$$

$$p(a,\theta) = \frac{\rho c^2 Q\, P_{\nu_o - 1/2}(-\cos\theta)(ka)^{8/3}}{4a^3 c_o} \qquad (38)$$

$$e^{\{-\alpha_o (ka)^{1/3}\pi - i\omega t + i\pi[ka + c_o (ka)^{1/2} + 2/3]\}} \qquad \theta = \pi$$

where

$$Q = \dot{W}\pi a^2$$
$$\Phi_m(\theta) = \frac{11\pi}{12} + ka[1 + c_m (ka)^{-2/3}]\,\theta - \omega t$$

B.) Transient Response

Consider the steepest-decent evaluation of the integral arising from

the high frequency end of the spectrum. For finite ω_o the factor

$$\frac{\omega}{\omega^2 - \omega_o^2}$$

does not affect the saddle-point value arising from the asymptotic evaluation of $\tilde{G}\,(a,\theta|a,0;\omega)$. The inverse may therefore be obtained by inverting the transform of the Green's Function and modifying the result by the above multiplicative factor evaluated at the resulting saddle-point value. The integral

$$\int_{-\infty}^{\infty} \tilde{G}\,(a,\theta|a,0;\omega)\, e^{-i\omega t}\, d\omega \tag{39}$$

is by definition of the inverse Fourier transform, 2π times the impulse Green's function $G\,(a,\theta,t|a,0;0)$.

Friedlander obtained an asymptotic expression for this Green's function, for $\theta \neq 0,\ \pi$ and for $\theta = \pi$ using one creeping-wave mode. Using consistent notation and expressing the Green's function in terms of the roots $(-\alpha_m)$ of the derivative of the Airy integral these expressions are

$$G(a,\theta;t|a,0;0) = \frac{-\theta}{4\pi\, 6^{1/2}\, a^2 (\sin\theta)^{1/2}} \left(\frac{a}{ct - a\theta}\right)^{3/2} H(ct - a\theta)\ \mathrm{X}$$
$$e^{-\left(\frac{2a}{ct - a\theta}\right)^{1/2} \left(\frac{\alpha_o \theta}{3}\right)^{3/2}} \qquad \theta \neq 0,\ \pi \tag{40}$$

$$G(a,\pi;t|a,0;0) = -\left(\frac{1}{2\pi^2\, 6^{5/4}\, a^2}\right) \left(\frac{a\pi}{ct - a\pi}\right)^{9/4} H\,(ct - a\pi)\ \mathrm{X}$$
$$\alpha_o^{3/4}\ e^{-\left(\frac{2a}{ct - a\pi}\right)^{1/2} \left(\frac{\alpha_o \pi}{3}\right)^{3/2}} \qquad \theta = \pi \tag{41}$$

As explained earlier a different formulation must be used for $\theta \approx 0$. To obtain the pressure close to the active element, Friedlander's result must be modified by retaining more than one "creeping-wave" mode. The number of modes required as a function of location θ and time will be discussed in a later section.

When the transient pressure is formulated as an inverse Laplace transform, the procedure consists of multiplying the terms in Equations 40 and 41 by the Laplace transform of Equation 34, evaluated at the saddle-point. Friedlander who formulates his impulse Green's function as an inverse Laplace rather than Fourier transform, gives the saddle-point value of the Laplace transform variable "s" as

$$\bar{s}_m = \left(\frac{a}{2c}\right)^{1/2} \left(\frac{\alpha_m \theta}{[3\ t-(a\theta/c)]}\right)^{3/2} \tag{42}$$

To use this result, we require the Laplace transform of Equation 34:

$$\tilde{W}(s) = \frac{\hat{W}\omega_o s}{s^2 + \omega_o^2} \tag{43}$$

Substituting the saddle-point value $s = s_m$, this becomes

$$\frac{W(\bar{s}_m)}{\hat{W}} = \frac{2^{1/2}[3(ct - a\theta)\,\alpha_m\theta]^{3/2}\ (k_o a)^2}{[54(\omega_o t - k_o\ a\theta)^3 + k_o a\ (\alpha_m\theta)^3]\ a^{3/2}} \tag{44}$$

Multiplying the terms in Equation 40 by this quantity, the transient pressure becomes

$$p(a,\theta;t) = \frac{3\rho c\dot{W}\alpha^2 \, (k_o a)^2 \theta^{5/2}}{4(\sin\theta)^{1/2}} \, H\left(t - \frac{a\theta}{c}\right) \quad X$$

$$\sum_{m=1} \frac{\alpha_m^{3/2} \exp\left[-\left(\frac{2a}{ct - a\theta}\right)^{1/2}\left(\frac{\alpha_m \theta}{3}\right)^{3/2}\right]}{54(\omega_o t - k_o a\theta)^3 + k_o a(\alpha_m \theta)^3} \tag{45}$$

The pressure is finite, even at the wavefront. However, at the center of the shadow zone, the focusing action is strong enough to generate a pressure singularity

$$p(a,\pi;t) = \frac{3^{1/4}\pi^{15/4}(k_o a)^2 \, \rho c\dot{W}\alpha^2 \frac{a^{3/4}}{(ct - a\pi)^{3/4}} H\left(t - \frac{a\pi}{c}\right)}{2^{7/4}} \quad X$$

$$\sum_{m=1} \frac{\alpha_m^{9/4}}{[54(\omega_o t - k_o a\theta)^3 + k_o a(\alpha_m \theta)^3]} \exp\left[-\left(\frac{2a}{ct - a\pi}\right)^{1/2}\left(\frac{\alpha_m \pi}{3}\right)^{3/2}\right] \tag{46}$$

6. Plane Baffle Analysis

As previously mentioned, the creeping-wave formulation does not yield the pressure on the active element and is inefficient in its immediate vicinity. In this and the proceeding sections the plane baffle analyses of Miles and Mangulis are presented for completeness, and shown to be applicable on and near the piston.

Miles[6] obtained the solution for the transient pressure on a circular piston embedded in an infinite plane baffle. The exciting pulse was assumed to be a Heaviside step function. Transform techniques were used and in this case a closed expression was available for the inverse transform.

$$p(r,0;t) = \begin{cases} \rho c\dot{W} & ct < (a - r) \\ \dfrac{\rho c\dot{W}}{\pi} \cos^{-1} \dfrac{c^2t^2 + r^2 - b^2}{2rct} & (r - b) \leq ct \leq (r + b) \\ 0 & (r + b) < ct < (r - b) \end{cases} \qquad (47)$$

where r is the radial distance measured on the plane of the baffle and b is the piston radius.

Mangulis[7] studied the case of a pulsed CW excitation and was able to obtain an expression for the resulting force on the piston

$$p(r,0;t) = \mathrm{Im}\{\dot{W}e^{i\omega_o t}[1- \frac{4}{\pi} \int_{\cos^{-1}\frac{ct}{2b}}^{\pi/2} \sin^2 \Theta\, e^{-i2\, k_o b \cos \Theta} d\Theta]\} \quad 0 \leq t \leq \frac{2b}{c}$$

$$(48)$$

$$p(r,0;t) = \mathrm{Im}\{\dot{W}e^{i\omega_o t}[1- \frac{J_1(2k_o b)}{k_o b} + \frac{iH_1(2k_o b)}{k_o b}]\} \quad t \geq 2b/c$$

where J_1 is a Bessel function, H_1 is a Struve function, and $k_o = \omega_o/c$.

7. Correction for Baffle Curvature on and Near Piston

The effect of the baffle curvature on the maximum pressure sensed by the piston and by the immediately adjacent baffle area elements can be shown to be non-existent for the former and small for the latter. Immediately after piston acceleration, when the entire piston senses the maximum pressure, $\rho c\dot{W}$, the pressure is confined to the piston. Therefore the piston "does not

know" whether it is in a plane or curved baffle, since the latter has not interacted with the shock wave. The immediately adjoining region of this baffle, senses its maximum pressure, $\rho c\dot{W}/2$, immediately after piston acceleration*, before more remote regions of the baffle have been reached by the shock wave. This pressure value should be affected to a negligible extent by baffle curvature, as long as the angle between the plane of the piston and the plane tangent to the baffle area element is extremely small.

It is only as the shock wave reaches more remote regions, where the shock front has been rotated through increasingly large angles, that baffle curvature may be of significance. It will now be shown by semi-quantitative reasoning that this effect is negligible in the entire range of θ where the plane baffle solution rather than the creeping-wave solution must be used.

If the empirical steady-state correction for baffle curvature is introduced in the inverse Laplace transform, analytical integration becomes impossible. The size of the correction which would be obtained if the

---

*More precisely the maximum pressure is

$$p_{max} = \frac{\rho c\dot{W}}{\pi} \cos^{-1} [(1 - \frac{a^2}{r^2})^{1/2}]$$

$$= \frac{\rho c\dot{W}}{2} \text{ as } r \to a$$

$$= \frac{\rho c\dot{W}}{\pi r} \text{ as } r >> a \ .$$

This maximum pressures occurs at time

$$t_m = \frac{1}{c} (r^2 - a^2)^{1/2}$$

The sudden drop from a maximum pressure $\rho c\dot{W}$ everywhere along the piston to half that value on the immediately adjoining baffle assumes no viscosity, i.e. an acoustic boundary layer of zero thickness.

baffle curvature was taken into account by evaluating the curvature-corrected inverse transform numerically will be seen to be negligible from the following consideration

The pressures on the spherical baffle and on the plane are approximately related as[12]

$$p_{sphere} \approx p_{plane} [1 - (ka)^{-1}] \tag{49}$$

The relevant value of ω, or ka, is the saddle-point value ω_1, Equation 23, corresponding to the time of maximum pressure (t_o), i.e.

$$t_o - \frac{a\theta}{c} = \frac{2a}{3^5 c} (|\alpha_m'|\theta)^3 \tag{50}$$

When this value of t is substituted in Equation 23, one finds

$$\overline{k_o a} = \frac{i3^6}{4} (|\alpha_m'|\theta)^{-3} \qquad \text{for } t = t_o \tag{51}$$

with θ in radians. This quantity is so large within the range of θ where the curvature-corrected plane-baffle solution is applicable as to make the correction term, Equation 49, negligible

8. Computation

The region of the (θ,t) domain in which the wave harmonic series (Equations 9 and 10) are applicable is limited by the difficulty in obtaining the values of $h_m''(z)$ evaluated at the (n + 1) complex roots of the equation $h_n'(z) = 0$. To extend the number of roots available beyond existing tabulated values (Table 1) may require the use of a computer subroutine which obtains the complex zeros of an arbitrary function. A qualitative

measure of the number of terms required as a function of position and time may be obtained from consideration of the stationary phase value of ka given by Equation 23. In the CW case, the extent of applicability of the series is also proportional to ω_o.

The creeping-wave series (Equation 28 and 45) for the transient pressure exhibit excellent convergence in its domain of applicability, i.e., in the vicinity of the wave fronts away from the active element. In most cases, a maximum of four terms has been found to satisfy a five precent convergence criterion. For $\theta >> \alpha$ only the first creeping-wave mode is required.

Equations 28 and 45 may also be used to evaluate the transient pressure when the active area can not be considered a point source. This situation would exist if a combination of elements were excited or if the creeping-wave formulation was desired in the immediate vicinity of the active area. In this case, Equations 28 and 45 may be considered as Green's functions and the resulting pressure obtained via a numerical integration over the active area. This is described in Figure 2.

9. Summary

The transient pressure on a spherical array arising from the excitation of a single circular element has been obtained. The cases of a Heaviside step and a continuous wave (CW) pulse were considered.

The short time transient effects were isolated into efficient creeping-wave series (Equations 28 and 45) while the long time results were obtained via classical wave harmonic analyses (Equations 9 and 10). In the case of a CW pulse, the short time transient effects were first separated from the steady-state response. The total solution thus is represented by the sum

of Equations 37 and 45 or 38 and 46.

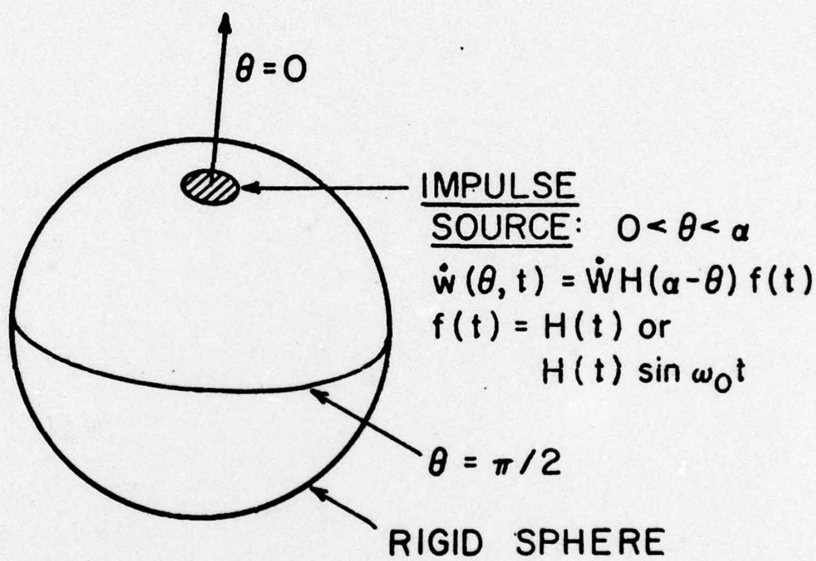
In the immediate vicinity of the active element, the Kirchhoff approximation has been shown to be applicable. Thus in this region the analyses of Miles and Mangulis (Equations 47 and 48) may be used. This completes the evaluation of the pressure in the relevant space and time domains.

Table 1

Second Derivative of the Spherical Hankel Function of order m Evaluated at the (m + 1) Roots of the Equation $h_m'(z) = 0$.

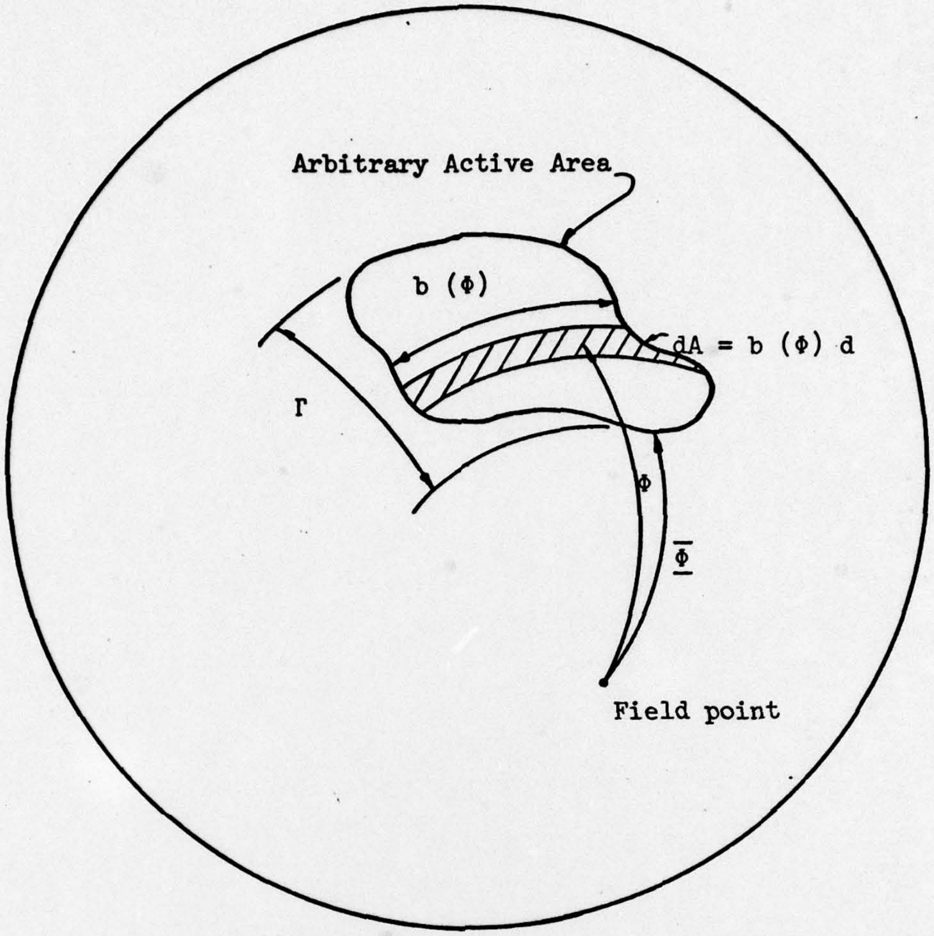
m	$h_m''(z_{mj})$
0	-2.718
1	$\mp .409 + i\,1.878$
2	2.516
	$.341 \pm i\,0.900$
3	$\pm 1.304 - i\,1.191$
	$\pm .339 + i\,0.458$
4	-1.5551
	$.2739 \pm i\,0.2661$
	$.4968 \mp i\,1.1320$

Figure 1



$\theta = 0$
IMPULSE
SOURCE: $0 < \theta < \alpha$
$\dot{w}(\theta, t) = \dot{W} H(\alpha - \theta) f(t)$
$f(t) = H(t)$ or
$H(t) \sin \omega_0 t$
$\theta = \pi/2$
RIGID SPHERE

Figure 2



$$p_{\text{distrib. source}}(a,\overline{\underline{\Phi}};t) = \int_{\text{Active area}} p_{\text{point source}}(a,\Phi;t)\, dA$$

$$= \int_{\overline{\underline{\Phi}}}^{\overline{b}+\Gamma} b(\Phi)\, p_{\text{point source}}(a,\Phi;t)\, d\Phi$$

where $p_{\text{point source}}(a,\Phi;t)$ is defined by either expression (28) or (45)

Φ represents the geodetic distance from field point to source point

b (Φ) is the width of the active element at Φ

D is the maximum length of the active area measured along the geodesic through the field point

$\overline{\underline{\Phi}}$ represents the shortest distance between the field point and the active area

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